Exhibit A

Nonlinear instabilities in magnetized plasmas: a geometrical treatment

Peter Dobias and John C. Samson

Abstract: The objectives of this paper are four-fold. The first, and main concern, is the development of an alternative approach to the description of plasma physics using methods of differential geometry. These methods have long been used in many other areas of physics, such as general relativity, or quantum field theory, but do not seem to have seen extensive application in plasma physics, and in particular in magnetohydrodynamics (MHD). The second objective is to employ this formalism for perturbation calculations, particularly to nonlinear processes in MHD. The use of differential geometry for variational calculations in ideal MHD allows a self-consistent, and compact calculation of the Lagrangian, and yields results valid for arbitrary topologies of the magnetic field. The third objective is to outline the use of this formalism in analyzing several plasma processes that occur in systems with complex magnetic-field topologies. We specifically focus on the nonlinear stability of plasmas in the magnetotail-like configuration of the magnetic field, such as found in the Earth's magnetosphere. Finally, we utilize previous results to present a self-consistent method for the investigation of the nonlinear stability of magnetized plasmas and for the investigation of the transition between linear and nonlinear behavior for systems close to equilibrium. This method is based on the analysis of potential energy density, using results for plasma displacement from a linear model to calculate the second- and third-order energies. We demonstrate this method on an example of a force-free field with magnetic-field lines stretched from dipolar configuration. In this example, we can clearly identify the transition between linear and nonlinear instability.

PACS Nos.: 52.30.–g, 52.35.–g

Résumé : Ce papier a quatre objectifs. Le premier et plus important concerne la possibilité de développer une approche alternative en physique des plasmas, qui utiliserait les méthodes de la géométrie différentielle. Ces méthodes sont utiles depuis longtemps dans d'autres domaines de la physique, comme la relativité générale ou la théorie quantique des champs, mais ne semblent pas avoir été utilisées en plasma, plus particulièrement en magnétohydrodynamique (MHD). Le deuxième objectif vise à utiliser ce formalisme pour effectuer des calculs perturbatifs, surtout touchant les phénomènes non linéaires en MHD. L'utilisation de la géométrie différentielle pour des calculs variationnels en MHD idéale permet un calcul auto-cohérent et compact du Lagrangien et donne des résultats valides pour des topologies arbitraires du champ magnétique. Le troisième objectif vise à développer l'utilisation de ce formalisme dans l'analyse de plusieurs processus qui se produisent dans des systèmes avec des topologies complexes de champ magnétique. Nous nous penchons surtout sur la stabilité

Received 28 October 2003. Accepted 16 April 2004. Published on the NRC Research Press Web site at http://cjp.nrc.ca/ on 13 July 2004.

P. Dobias[1] and J.C. Samson. Department of Physics, University of Alberta, Edmonton, AB T6G 2J1, Canada.
[1] Corresponding author (e-mail: pdobias@space.ualberta.ca).

 doi: 10.1139/P04-033

non-linéaire de plasmas en configuration de queue magnétique, comme dans le cas de la magnétosphère terrestre. Finalement nous utilisons de précédents résultats pour présenter une méthode auto-cohérente d'analyse de la stabilité non linéaire du plasma et de la transition entre les comportements linéaire et non linéaire pour des systèmes près de l'équilibre. La méthode se base sur l'analyse de la densité d'énergie potentielle, utilisant des résultats de déplacement du plasma provenant d'un modèle linéaire pour calculer les énergies aux deuxième et troisième ordres. Nous démontrons la méthode dans un exemple de champ libre avec des lignes de champ dipolaire étirées. Dans cet exemple, nous voyons clairement la transition entre les instabilités linéaires et non linéaires.

[Traduit par la Rédaction]

1. Introduction

The problem of nonlinear plasma instabilities is of a great importance in many areas of space and laboratory plasma physics. These instabilities have many common characteristics even though the configurations of the plasmas might be quite different [1], and are characterized by [2]

- sudden onset,
- burst of energy, momentum, and particles across magnetic surfaces,
- avalanche character (localized instability can disrupt the whole system [3]).

For example, in space physics, observations show that during the onset of the intensification of a magnetic substorm a huge amount of energy is suddenly released on time scales of the order of minutes (Alfvénic time scale) [4]. These time scales suggest that the processes involved are highly nonlinear (explosive). There were several previous attempts to create a working model of explosive instabilities in magnetized fluids [3, 5]. Approaches based on perturbation theory in the framework of magnetohydrodynamics (MHD) typically lead to extremely complicated expressions.

A very practical way to address the question of nonlinear stabilities appears to be a variational approach [6–8]. The advantage of a variational approach is that one is dealing with a scalar quantity (energy), which makes this formalism suitable in any magnetic-field geometry.

We present an alternative mathematical approach to the Lagrangian–Hamiltonian method, employing differential geometry [9–11], using tensor notation combined with general transformation rules for the change of coordinates. Further, we use the generalized Stokes theorem to derive equilibrium conditions and wave equations from the expansion of the Lagrangian. This approach is very effective in eliminating redundant terms, and in minimizing the complexity of the algebra. In addition, the results are independent on the topology of the magnetic field.

Our derivation follows the general method for the expansion of the Lagrangian of ideal MHD outlined in ref. 8. Treating the plasma displacement as a coordinate change, we derive expressions for the magnetic field, plasma pressure, and density up to the third order in plasma displacement. Then we express the ideal MHD Lagrangian up to the third order in plasma displacement using these expressions. In the third-order term, we correct an error in the contribution of the magnetic field in ref. 8. We then derive equilibrium conditions and general wave equations for plasmas without ambient convection from the first- and second-order Lagrangian term, respectively. This linear equation makes possible calculations of plasma eigenmodes in complicated topologies of the magnetic field, since it is derived with no restriction on the topology of the field. Also, we present several suggestions on the use of this formalism in the investigation of nonlinear plasma instabilities. We also show the consistency of our results with the established works on two special configurations of the plasma.

Further, we apply certain restrictions to the wave equation and derive the equations for MHD field-line resonances [12] and linear ballooning modes [13], and we demonstrate that these results are compatible with published results. These examples allow us to illustrate the strengths of our approach

and make a direct connection between our results and those using vector formalisms. However, we show that our method is self-consistent and much more transparent than the vector approaches.

Finally, we present a model that investigates whether certain configurations of magnetized plasmas can lead to an onset of explosive behavior. Our method is based on a comparison of the second- and third-order potential energies for the plasma displacement obtained as a solution of the linear equation. The assumption behind this method is that the system we are investigating is not far from equilibrium, and that initially linear behavior, whether stable or unstable, is dominant. During the temporal evolution, however, the system can undergo a transition from linear to nonlinear behavior. We demonstrate the use of this method with an example of a magnetic-field topology that is stretched from the dipolar configuration, similar to the stretching in the magnetotail during the growth phase of the magnetospheric substorm [14]. Since the stretching of the field lines provides a source of free energy in the system, due to enhanced currents [1], evolution of an instability might be expected in this system. The model we will present here does not attempt to describe the full dynamics of the instabilities, since these dynamics are beyond the scope of ideal MHD, which assumes sufficiently slow processes and large spatial scales.

Our work has four objectives. The first is to develop a formalism that will simplify what are now very complicated calculations using variational methods in plasma physics. In this regard the methods of differential geometry have been very effective in other areas of physics where complex topologies might arise (see, for example refs. 10 and 15). Geometrical methods have also been used in connection with plasma physics [16, 17]. We then address two further objectives, the correction of the results in ref. 8, and an outline of methods to solve some problems in plasmas with complicated magnetic-field topologies. Last, we present an example of the nonlinear stability of a cold plasma in a stretched field topology.

2. Lagrangian in ideal MHD

Although ideal MHD gives one of the simplest approximations of plasma dynamics, it still describes some very interesting physical features of the plasma. This relative simplicity gives us an ideal tool for developing new methods for studying the behavior of plasmas. In ideal MHD, the Lagrangian has the simple form (the notation is given in the Appendix)

$$L = \int \mathrm{d}V \left(\frac{1}{2}\rho v^2 - \frac{p}{\gamma - 1} - \frac{1}{2}B^2 \right) \tag{1}$$

This Lagrangian needs constraints to yield nontrivial behavior. The constraints we use are mass conservation $\int_V \rho\, \mathrm{d}V = \text{const}$, magnetic flux conservation $\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \text{const}$ and the adiabatic condition $p\rho^{-\gamma} = \text{const}$.

We treat plasma perturbations as a coordinate change

$$\hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t) \tag{2}$$

where the displacement $\boldsymbol{\xi}$ is a function of the original coordinates $\boldsymbol{x}$, and time t. Also, we assume that the displacement ξ and its gradients ($\xi^i{}_{,j}$) are small enough so that we can use them as the ordering quantities for a perturbation treatment of the plasma. Then the whole perturbation calculus can be done in a geometric framework, without introducing any new physical constraints on the system. The geometrical approach uses a component formalism, combined with the use of definitions for the transformation of the volume and surface (instead of deriving them through additional physical assumptions [8], and uses geometrical definitions of determinants and inverse matrices during the derivation process. This formalism simplifies the algebra significantly, and the results are in a compact form that is suitable for further calculations. It also has the advantage that the results do not depend on the coordinate system. Consequently, the results hold for any geometry, and do not limit the physics beyond limitations introduced by use of ideal MHD.

2.1. Transformation of kinetic energy

The perturbed velocity has the same form $\hat{v} = \mathrm{d}\hat{x}/\mathrm{d}t$ as the unperturbed velocity, with the difference that the plasma displacement is also an explicit function of x and, therefore, total differentiation is necessary. Taking into account mass conservation, and the fact that $\mathrm{d}\hat{V} = J\mathrm{d}V$, which implies that $\hat{\rho} = \rho/J$, where J is the Jacobian of the coordinate change (2), the perturbed kinetic energy has the form

$$\hat{K} = \int \mathrm{d}V \frac{1}{2}\rho\left(v + \partial_t\xi + v^j\partial_j\xi\right)^2 \tag{3}$$

In the absence of ambient convection the first and the third terms in brackets vanish and the kinetic energy is simply

$$\hat{K} = \int \mathrm{d}V \frac{1}{2}\rho\,(\partial_t\xi)^2 \tag{4}$$

This will be the form for the kinetic energy used throughout the rest of the work.

2.2. Transformation of thermal and magnetic energy

Since mass conservation combined with the rule for the transformation of a volume element yields $\hat{\rho} = \rho/J$, the application of the adiabatic condition, connecting density and pressure yields the transformation relation for the pressure in the form $\hat{p} = pJ^{-\gamma}$.

Similarly, since the surface element $\mathrm{d}S$ transforms as $\mathrm{d}\hat{S}_i = A_i^j\mathrm{d}S_j$, as will be shown in the next section together with the derivation of the transformation matrix A, magnetic flux conservation yields $\hat{B}^i = (A^{-1})^i_j B^j$. So the potential energy terms are

$$\hat{E}_T = \int \mathrm{d}V \frac{pJ^{1-\gamma}}{\gamma - 1} \tag{5}$$

for the thermal component, and

$$\hat{E}_M = \int \mathrm{d}V \frac{1}{2} J g_{ij}\left(A^{-1}\right)^i_k B^k \left(A^{-1}\right)^j_l B^l \tag{6}$$

where g_{ij} is a metric tensor, for the magnetic energy. Therefore, combining (4) with (5) and (6), the perturbed Lagrangian in the absence of convection is

$$\hat{L} = \int \mathrm{d}V\left(\frac{1}{2}\rho(\partial_t\xi)^2 - \frac{pJ^{1-\gamma}}{\gamma - 1} - \frac{1}{2}Jg_{ij}\left(A^{-1}\right)^i_k B^k\left(A^{-1}\right)^j_l B^l\right) \tag{7}$$

So far our results are consistent with those derived in ref. 8. However, our method of derivation of the Jacobian and the matrix A differs significantly. The method we will present in the next section is self consistent, purely geometrical, and does not require any additional physical assumptions, for instance, restrictions on the magnetic field. The use of the component formalism and the use of generalized Kronecker deltas and Levi–Civita symbols ensures a very compact form for the results, making further calculations much simpler. Also, the authors of ref. 8 made the error of assuming that the inverse matrix A^{-1} is second order. Although this is true for the matrix A, it is not true for the matrix A^{-1}. Due to the omission of these terms, their expression for the third-order perturbation in the Lagrangian cannot be generally correct. In the special case they discussed, with the trivial third-order term, these additional terms vanish as well. At the end of this section, we present a more detailed analysis of the terms that ref. 8 omitted due to the missing third-order term in the expansion of A^{-1}. Another important point is that a major advantage of the formalism we use occurs in situations with special symmetries. Many terms disappear due to their symmetry properties, significantly reducing the complexity of calculations.

2.3. Transformation of volume and surface

Starting with the definition of the Jacobian $J = \det(\hat{x}^i_{,k})$ and using the expression for the determinant of an $n \times n$ matrix A, $\det A^i_k = \delta(n)A \cdots A$ [9, 11], the expression

$$J = \delta^{ijk}_{lmn}\left(\delta^l_i + \xi^l_{,i}\right)\left(\delta^m_j + \xi^m_{,j}\right)\left(\delta^n_k + \xi^n_{,k}\right) = 1 + \xi^i_{,i} + \xi^i_{[,i}\xi^j_{,j]} + \xi^i_{[,i}\xi^j_{,j}\xi^k_{,k]} \tag{8}$$

for the Jacobian is obtained.

The surface element is defined as $\mathrm{d}S_l = \epsilon_{ijk}\,\mathrm{d}x^j\,\mathrm{d}x^k$. Using this definition for the perturbed surface, noting that $\mathrm{d}\hat{x}^j = \left(\delta^j_k + \xi^j_{,k}\right)\mathrm{d}x^k$, and using the fact that the product of symmetric and antisymmetric tensors is zero, the expression

$$\mathrm{d}\hat{S}_i = 3\delta^{lmn}_{ijk}\left(\delta^j_m + \xi^j_{,m}\right)\left(\delta^k_n + \xi^k_{,n}\right)\epsilon_{lop}\,\mathrm{d}x^o\,\mathrm{d}x^p \equiv A^l_i\,\mathrm{d}S_l \tag{9}$$

for the transformation of surface elements is obtained. This also defines the transformation matrix A used in the previous section. However, for the calculation of the magnetic energy density, the inverse matrix A^{-1} is needed. This inverse matrix has a form similar to the original matrix A, and

$$\left(A^{-1}\right)^i_j = 3\delta^{ikl}_{jmn}\left(\delta^m_k + \xi^m_{,k}\right)^{-1}\left(\delta^n_l + \xi^n_{,l}\right)^{-1} \tag{10}$$

The inverse matrix $(J^{-1})^m_k \equiv \left(\delta^m_k + \xi^m_{,k}\right)^{-1}$ is by definition expressed as $\left(J^{-1}\right)^i_j \equiv (3/J)\delta^{ikl}_{jmn}J^m_k J^n_l$. Expanded to the third order it is

$$\left(J^{-1}\right)^i_j = \delta^i_j - \xi^i_j + \xi^i_l\xi^l_j - \xi^i_l\xi^l_k\xi^k_j + \mathrm{O}\left(\xi^4\right) \tag{11}$$

Substituting (11) into (10) and expanding the result to the desired order one obtains the inverse matrix A^{-1} with arbitrary accuracy. Expansion up to the third order yields

$$\begin{aligned}\left(A^{-1}\right)^i_j = \delta^i_j + \xi^i_{,j} - \delta^i_j\xi^a_{,a} - \xi^i_{,j}\xi^a_{,a} + \frac{1}{2}\delta^i_j\left(\xi^a_{,a}\xi^b_{,b} + \xi^a_{,b}\xi^b_{,a}\right)\\ - \delta^i_j\xi^a_{,a}\xi^b_{,c}\xi^c_{,b} + \xi^i_{,j}\xi^a_{,b}\xi^b_{,a} + \xi^i_{,a}\xi^a_{,j}\xi^b_{,b} - \xi^i_{,a}\xi^a_{,b}\xi^b_{,j} + \mathrm{O}\left(\xi^4\right)\end{aligned} \tag{12}$$

Equivalent results for the transformation of the magnetic field could be obtained in this special case by using the Lundquist identity, which can be written in the form $\hat{B}^i = B^j\left(\delta^i_j + \xi^i_{,j}\right)/J$ [18]. Instead, we derived this relation directly from the magnetic flux conservation to demonstrate this method in its most general sense, with the direct use of the constraint to Lagrangian (1).

2.4. Perturbed Lagrangian up to the third order in plasma displacement

Now it is possible to substitute expressions for the Jacobian (8) and A^{-1} (12) into expressions for the thermal and magnetic energy (5) and (6).

2.4.1. Thermal energy

Substituting expression (8) into expression (5) and using a Taylor expansion of the form $(1+J)^{1-\gamma} = 1 + (1-\gamma)J - (1/2)\gamma(1-\gamma)J^2 + (1/6)\gamma(1-\gamma^2)J^3$ in (5) one obtains the expression for the thermal component of the potential energy up to the third order

$$\begin{aligned}\hat{E}_\mathrm{T} = \int \mathrm{d}V\frac{p}{\gamma-1}\Big[1 + (1-\gamma)\left(\xi^i_{,i} + \xi^{[i}_{,i}\xi^{j]}_{,j} + \xi^{[i}_{,i}\xi^j_{,j}\xi^{k]}_{,k}\right)\\ + \frac{\gamma(\gamma-1)}{2}\left(\xi^i_{,i}\xi^j_{,j} + 2\xi^i_{,i}\xi^{[j}_{,j}\xi^{k]}_{,k}\right) + \frac{\gamma\left(1-\gamma^2\right)}{6}\xi^i_{,i}\xi^j_{,j}\xi^k_{,k}\Big]\end{aligned} \tag{13}$$

2.4.2. *Magnetic energy*

After substituting expressions (8) and (12) into expression (6) one obtains for the magnetic energy

$$\hat{E}_{\mathrm{M}} = \int \mathrm{d}V \frac{1}{2} \boldsymbol{B}^2 \left(1 - \xi^i_{,i} + \frac{1}{2}\left(\xi^i_{,i}\xi^j_{,j} + \xi^i_{,j}\xi^j_{,i}\right) + \xi^i_{[,i}\xi^j_{,j}\xi^k_{,k]} + \xi^i_{,i}\xi^j_{,k}\xi^k_{,j}\right) + \frac{1}{2} g_{ij} B^k B^l \xi^i_{,k}\xi^j_{,l}\left(1 - \xi^m_{,m}\right)$$
$$+ g_{ij} B^i B^l \left(\xi^j_{,l}\left(1 - \xi^k_{,k} + \xi^k_{,m}\xi^m_{,k}\right) + \xi^j_{,k}\xi^k_{,l}\xi^m_{,m} - \xi^j_{,k}\xi^k_{,m}\xi^m_{,l}\right) \quad (14)$$

The magnetic energy density consists of two components, the first one is magnetic pressure and the second one takes into account curvature. In a typical vector formalism for deriving (13) and (14), there is often a very large number of terms [8], and it is often not clear which terms are zero and which terms cancel. Our use of component notation eliminates this problem, giving far fewer terms, and equations that are much easier to use in the analysis of stability problems.

2.5. Expansion of the Lagrangian

Using expressions (4), (13), and (14) it is possible to expand Lagrangian (7) with respect to the plasma displacement as

$$\hat{L} = \int \mathrm{d}V \left(\mathcal{L}^0 + \mathcal{L}^1(\xi) + \mathcal{L}^2(\xi^2) + \mathcal{L}^3(\xi^3) + \cdots\right) \quad (15)$$

The terms $\mathcal{L}^0$ to $\mathcal{L}^3$ are

$$\mathcal{L}^{(0)} = -\frac{p}{\gamma - 1} - \frac{1}{2}\boldsymbol{B}^2 \quad (16)$$

$$\mathcal{L}^{(1)} = \left(p + \frac{1}{2}\boldsymbol{B}^2\right)\xi^k_{,k} - g_{ij} B^i B^l \xi^j_{,l} \quad (17)$$

$$\mathcal{L}^{(2)} = \frac{1}{2}\Big[\rho(\partial_t \xi)^2 - p\left((\gamma - 1)\xi^a_{,a}\xi^b_{,b} + \xi^a_{,b}\xi^b_{,a}\right) - g_{ij} B^k B^l \xi^i_{,k}\xi^j_{,l} + 2 g_{ij} B^i B^l \xi^j_{,l}\xi^k_{,k}$$
$$- \frac{1}{2}\boldsymbol{B}^2\left(\xi^a_{,a}\xi^b_{,b} + \xi^a_{,b}\xi^b_{,a}\right)\Big] \quad (18)$$

$$\mathcal{L}^{(3)} = p\left(\xi^{[a}_{,a}\xi^b_{,b}\xi^{c]}_{,c} - \gamma\xi^a_{,a}\xi^{[b}_{,b}\xi^{c]}_{,c} + \frac{\gamma(1 + \gamma)}{6}\xi^a_{,a}\xi^b_{,b}\xi^c_{,c}\right) - \frac{1}{2}\boldsymbol{B}^2\left(\xi^{[a}_{,a}\xi^b_{,b}\xi^{c]}_{,c} + \xi^a_{,a}\xi^b_{,c}\xi^c_{,b}\right)$$
$$+ \frac{1}{2} g_{ij} B^k B^l \xi^i_{,k}\xi^j_{,l}\xi^a_{,a} - g_{ij} B^i B^l \left(\xi^j_{,l}\xi^a_{,b}\xi^b_{,a} + \xi^j_{,a}\xi^a_{,l}\xi^b_{,b} - \xi^j_{,a}\xi^a_{,b}\xi^b_{,l}\right) \quad (19)$$

The zeroth-order term is simply the unperturbed Lagrangian of ideal MHD. Applying the variational condition $\int \mathrm{d}t \int \mathrm{d}V \mathcal{L}^{(i)} = 0$ to the first- and second-order terms, it is possible to derive the equilibrium condition and momentum equation, respectively. In the next section, we will discuss the linear wave equation in more detail, and we will show consistency between our results and other established works. The third-order term contains information about possible explosive behavior [8]. This is discussed in more details later.

2.6. Effect of the third-order contribution in transformation of magnetic field

As was mentioned earlier, the authors of ref. 8 omitted the third-order term in the derivation of the matrix A^{-1} that is used in the transformation of the magnetic field. They used this matrix as an intermediate step in their derivation of the matrix A, which is indeed of the second order in plasma displacement. However, A^{-1} also contains higher order expansion terms and these terms become important in the transformation of the magnetic field.

Since a direct comparison between the expression from the Lagrangian obtained in ref. 8 with expression (15) is very difficult to perform due to the complexity of their result (containing a large number of redundant terms), we decided instead to trace the difference that omission and (or) inclusion of the third-order term in the expansion of A^{-1} causes in the final expansion of the magnetic energy.

A^{-1} appears in the magnetic energy in the expression $g_{ij}\left(A^{-1}\right)^i_k\left(A^{-1}\right)^j_l B^k B^l$. The magnetic pressure term in ref. 8 would be (3/2) $\left(B^2\xi^a_{,a}\xi^b_{,c}\xi^c_{,b}\right)$, whereas our term is (1/2) $\left(B^2\xi^a_{,a}\xi^b_{,c}\xi^c_{,b}\right)$. This introduces an asymmetry between thermal and magnetic pressure terms in Pfirsch and Sudan's results. This will affect any system with nonvanishing $\nabla\cdot\xi$.

All the other terms affect the curvature part of the magnetic energy. In fact, the only curvature term that remains with no third-order term in the A^{-1} matrix is simply (1/2) $\left(g_{ij}B^k B^l\xi^a_{,a}\xi^i_{,k}\xi^j_{,l}\right)$ and the whole expression $g_{ij}B^i B^l\left(\xi^j_{,l}\xi^b_{,c}\xi^c_{,b}+\xi^j_{,a}\xi^a_{,l}\xi^b_{,b}-\xi^j_{,a}\xi^a_{,b}\xi^b_{,l}\right)$ is omitted. This means that any system with the plasma displacement changing in any direction other than along the field lines will not be described correctly. Again, in the situations with special symmetry that the authors of ref. 8 use to verify their result, the third-order term is identically zero due to the symmetry properties. In these special instances their results and our results agree.

3. Variational calculus

Using a variational approach in the second (17) and the third term (18) of the Lagrangian (15), it is possible to derive equations governing the dynamics of the plasma in a linear approximation. The condition $\int \mathrm{d}V\mathcal{L}^{(1)}=0$ yields the equilibrium condition in the form

$$-\left(p+\frac{1}{2}B^2\right)_{,i}+g_{il}B^j B^l_{,j}=0 \tag{20}$$

This is the standard equation containing the sum of thermal and magnetic pressure and pressure due to field-line curvature.

The condition $\int \mathrm{d}t\int \mathrm{d}V\mathcal{L}^{(2)}=0$ yields the linearized momentum equation for plasmas in the form

$$\begin{aligned}\rho\partial_{tt}\xi_i=\partial_i&\left(\gamma P\frac{1}{\sqrt{g}}\partial_l\sqrt{g}\xi^l+\xi^l\partial_l P-g_{lk}B^k\frac{1}{\sqrt{g}}\partial_p\sqrt{g}\left(\xi^l B^p-\xi^p B^l\right)\right)\\&+B^l\partial_l\left(\frac{g_{ij}}{\sqrt{g}}\partial_p\sqrt{g}\left(\xi^j B^p-\xi^p B^j\right)\right)-\left(\partial_i B^l\right)\frac{g_{lj}}{\sqrt{g}}\partial_p\sqrt{g}\left(\xi^j B^p-\xi^p B^j\right)\\&+\left(\partial_i B_l-\partial_l B_i\right)\left(\frac{1}{\sqrt{g}}\partial_p\sqrt{g}\left(\xi^l B^p-\xi^p B^l\right)\right)\end{aligned} \tag{21}$$

Obviously (21) is a wave equation describing linear wave propagation in a plasma with arbitrary (and possibly very complex) magnetic-field topology, for which ideal MHD is a valid approximation. There are no further restrictions on the direction of propagation nor on the configuration of the ambient quantities. The only restriction on the validity of this equation is that the wave amplitude be small enough for the linear approximation to be valid. However, by applying restrictions on this equation, it is possible to derive dispersion relations for waves in specific plasma configurations. Since the methods of

differential geometry are not commonly used in MHD, in the following section we give two examples (field-line resonances in a rectilinear field topology and ballooning modes in a curvilinear field) showing that equation (21) yields results consistent with established results using standard vector methods, but in a more transparent way.

3.1. Field-line resonances (FLRs)

In the case of a rectilinear magnetic field the natural choice for coordinates is a Cartesian coordinate system. Then the metric tensor $g_{ij} = \mathrm{diag}(1, 1, 1)$, and vectors and covectors are identical. The equilibrium condition (20) takes the simple form $(P + B^2/2)_{,i} = 0$. Using this in equation (21), and assuming a harmonic time dependence, the momentum equation has the form

$$\rho\omega^2\xi^i = -\partial^i\left(\left(\gamma P + B^2\right)\partial_l\xi^l - B_j B^p\partial_p\xi^j\right) - B^l\partial_l\partial_p\left(\xi^i B^p - \xi^p B^i\right) - \left(\partial_j B^i\right)\partial_p\left(\xi^j B^p - \xi^p B^j\right) \quad (22)$$

We now choose axes such that the z axis is along the magnetic field lines, and the variation of the ambient quantities is along the x axis. The magnetic field is $B = (0, 0, B)$ and all derivatives of P, B, and ρ except ∂_x vanish. Also assuming that the displacement is harmonic along the y and z directions

$$\xi(\rho\partial_{tt}\xi; x) = \xi(x)\exp i(k_y y + k_z z) \quad (23)$$

equation (22) can be reduced to

$$\rho\left(\omega^2 - k_z^2 V_A^2\right)\xi_x = -\partial_x\left(F(x)\partial_x\xi_x\right) \quad (24)$$

where F is defined as

$$F(x) = \frac{\rho\left(\omega^2 - k_z^2V_A^2\right)\left(\omega^2V^2 - V_A^2V_S^2k_z^2\right)}{\omega^2\left(\omega^2 - V^2k^2\right) + k_z^2k^2V_A^2V_S^2} \quad (25)$$

$$V_A^2 = \frac{B^2}{\rho}, \qquad V_S^2 = \frac{\gamma P}{\rho}, \qquad V^2 = V_A^2 + V_S^2, \qquad k^2 = k_y^2 + k_z^2$$

This result is consistent with the expression obtained in ref. 12. Equation (24) yields two turning points and two resonances. In the case of a cold plasma, expression (25) reduces to

$$F(x) = \frac{\rho V_A^2\left(\omega^2 - k_z^2V_A^2\right)}{\omega^2 - \left(k_z^2 + k_y^2\right)V_A^2} \quad (26)$$

This equation has only one turning point at $\omega^2 = k^2V_A^2$ and one resonance at $\omega^2 = k_z^2V_A^2$. Occurrence of this resonance is due to coupling between the compressional Alfvén wave and shear Alfvén wave of the same frequency. Since there is no curvature, ballooning modes cannot evolve and thus this mode cannot be explosively unstable. In the last part of this paper, we will show energy transfer in this mode as part of the demonstration of the stability investigation.

Though we presented a simple model of FLRs in a Cartesian topology, the strength of our method will be more evident in complex magnetic-field topologies. One example of more complex topologies occurs in the region of stretched field lines in the near-Earth magnetotail [14] during the substorm growth phase. Though these modes play an extremely important role in the plasma dynamics of the magnetosphere, we know of no complete solutions (analytic or numerical) of the equations for the eigenmodes. The linear wave equation (21) is written in a form that is suitable for such a calculation. All that needs to be defined is the metric tensor of the topology we are working in. If this metric is connected to the magnetic field, then all the information we need is contained in this tensor.

3.2. Linear ballooning mode

We consider a second example with a curved magnetic field and an orthonormal coordinate system defined by

$$\hat{e} = \frac{B}{B} \tag{27}$$

$$\hat{n} = R_c \hat{e} \cdot \nabla \hat{e} \tag{28}$$

$$\hat{\phi} = \hat{e} \times \hat{n} \tag{29}$$

Here again, the metric is $g_{ij} = \mathrm{diag}(1, 1, 1)$, and vectors and covectors are identical. Introducing the definitions

$$\kappa_p = \hat{n} \cdot \nabla \ln P \tag{30}$$

$$\kappa_b = \hat{n} \cdot \nabla \ln B \tag{31}$$

$$\kappa_c = \hat{n} \cdot (\hat{e} \cdot \nabla)\hat{e} \tag{32}$$

the equilibrium condition (20) can be written in components as

$$\partial_\| P = 0 \tag{33}$$

$$\frac{V_S^2}{\gamma V_A^2}\kappa_p = \kappa_c - \kappa_b \tag{34}$$

$$\partial_\phi P = 0 \tag{35}$$

Assuming no azimuthal dependence ($\partial_\phi = 0$) of the ambient field, the problem reduces to two dimensions. Equation (21) can be split into components in the parallel and radial directions. Assuming that the ambient magnetic field changes slowly along the field lines, and consequently assuming a harmonic dependence of the displacement along the field lines (i.e., $\partial_\| \rightarrow ik_\|$), and in time (i.e., $\partial_t \rightarrow -i\omega$), the parallel component of equation (21) is simplified to

$$\xi_\| = -ik_\| \frac{V_S^2}{\omega^2} \partial_l \xi^l \tag{36}$$

Subsequently, assuming very small-scale sizes in the azimuthal direction ($k_\phi \rightarrow \infty$), which is equivalent to averaging the total perturbation pressure $P_T = \delta p + B \cdot \delta B$ to zero [13], which in the terms of equation (21) is

$$P_T \equiv \gamma P \partial_l \xi^l + \xi^l \partial_l P - B_l \partial_p \left(\xi^l B^p - \xi^p B^l\right) = 0 \tag{37}$$

and using (36), the equation

$$\left(V_S^2 + V_A^2 - \frac{V_A^2 V_S^2 k_\|^2}{\omega^2}\right) \partial_j \xi^j = -2\kappa_c V_A^2 \xi_n \tag{38}$$

connecting the compressional and shear Alfvén wave, is obtained.

In the absence of the perturbation pressure, the radial component of equation (21) is simply

$$\left[\omega^2 - V_A^2 k_\|^2 - 2V_A^2 \kappa_c(\kappa_b + \kappa_c)\right]\xi_n = 2\kappa_c V_A^2 \left(1 - \frac{V_S^2 k_\|^2}{\omega^2}\right) \partial_j \xi^j \tag{39}$$

This equation describes the generation of the shear Alfvén wave by the compressional wave. Equations (38) and (39) describe a linear ballooning mode, and are consistent with results obtained in ref. 13.

3.3. Cold plasma approximation

As the last example, we shall derive an instability criterion in the cold plasma approximation. This approximation assumes plasma pressure $P = 0$. Then, using an orthonormal coordinate system with coordinate x_1 along the magnetic field, the wave equation (21) reduces to the form

$$\rho \partial_{tt} \xi_1 = 0 \tag{40}$$

$$\rho \partial_{tt} \xi_2 = \frac{B}{h_1 h_2} \left[\partial_2 \left(\partial_2 B h_3 \xi_2 + \partial_3 B h_2 \xi_3 \right) + \partial_1 \left(\frac{h_2}{h_1 h_3} \partial_1 B h_3 \xi_2 \right) \right] \tag{41}$$

$$\rho \partial_{tt} \xi_3 = \frac{B}{h_1 h_3} \left[\partial_3 \left(\partial_2 B h_3 \xi_2 + \partial_3 B h_2 \xi_3 \right) + \partial_1 \left(\frac{h_3}{h_1 h_2} \partial_1 B h_2 \xi_3 \right) \right] \tag{42}$$

where h_i are the Lame coefficients (square roots of the diagonal elements of the metric tensor). Since the parallel displacement ξ_1 is constant, we can generally assume it is equal to zero and the problem reduces to two equations. Assuming a harmonic temporal and spatial dependence of the displacement, with infinite azimuthal wavelength, and assuming that the ambient quantities vary much more slowly than the plasma displacement we obtain the dispersion relation for the plasma

$$\omega^4 - V_A^2 \omega^2 \left(2\frac{k_1^2}{h_1^2} - \frac{k_2^2}{h_2^2} \right) + V_A^4 \frac{k_1^2}{h_1^2} \left(\frac{k_1^2}{h_1^2} - \frac{k_2^2}{h_2^2} \right) = 0 \tag{43}$$

Here $V_A^2 = B^2/\rho$, h_1, and h_2 are Lame coefficients with coordinates x_1 and x_2, and k_1 and k_2 are wave vectors in directions x_1 and x_2, respectively. Equation (43) yields two solutions for ω^2

$$\omega_1^2 = V_A^2 \left(\frac{k_1^2}{h_1^2} - \frac{k_2^2}{h_2^2} \right) \tag{44}$$

$$\omega_2^2 = V_A^2 \frac{k_1^2}{h_1^2} \tag{45}$$

This solution allows for instability if

$$\frac{k_1^2}{h_1^2} < \frac{k_2^2}{h_2^2} \tag{46}$$

In the magnetosphere-like configurations, the wavelength along the field line is often much larger than the radial scale size, these systems can easily yield unstable behavior. More detailed discussion of instabilities in force-free field can be found in ref. 1.

4. Explosive instabilities

4.1. Outline of the method

The third-order term in the Lagrangian (15) can, under certain circumstances, provide us with the information about the nonlinear behavior of the plasma, without the need to understand the nonlinear dynamics of the system. There are some instances (some of the configurations with special symmetries that have the third-order energy identically zero) when this method will not work. We will not be considering these special configurations, instead we assume that the expansion (15) contains all the terms.

The method we outlined is based on the observation that equations like

$$\ddot{x} \propto x^2 \tag{47}$$

which correspond to a mechanical system with the potential energy $U \propto x^3$, have singular solutions of the form

$$x \propto \frac{1}{(t-t_0)^2} \tag{48}$$

Therefore, we assume that systems with dominant third-order potential energy (or, in other words, a dominant quadratic term in the equation of motion) exhibit singular behavior. If during the evolution of the system the third-order potential energy starts to dominate, we are looking at the onset of explosive behavior. This idea was outlined in ref. 8, however, the complexity of the expressions makes it very difficult to apply this idea to specific problems. The presence of higher order terms can possibly lead to the saturation of the instability [19]. This is, however, beyond the scope of our discussion here. Our goal is to identify the onset of the instability, not to resolve the full dynamics of the instability.

4.2. Instability investigation

As we have outlined, analysis of the energy can give us stability properties of the plasma in the nonlinear regime without having to solve nonlinear equations. The whole process can be divided into three steps.

First, it is necessary to define the equilibrium or quasi-equilibrium configuration of the plasma that we want to investigate. This will define our ambient pressure, magnetic field, and plasma density P, $\boldsymbol{B}$, and ρ. The magnetic field, $\boldsymbol{B}$, then defines the geometry of the problem.

For the second step, we solve the linear wave equation (21) to obtain the class of allowed displacements for a given configuration. Obviously, it would be useless to investigate all displacements, since most of them would be nonphysical anyway. We assume that the solution of the linearized equation of motion is a good approximation of the real displacement, since we do not attempt to describe the nonlinear stage, we only want to predict its beginning. This approximation is justified a posteriori by the dominance of the second-order energy at the initial stage of the model. At this stage it is also possible to derive the linear growth rate and thus the linear stability criterion.

In the third step, we evaluate the second- and third-order potential energies and compare them. When the third-order term starts to dominate, the system has reached the threshold of explosively unstable behavior. If the second-order term dominates, linear behavior prevails. This linear behavior is already resolved in step two. During the stage when the second- and third-order terms are comparable, we are looking at a transition stage with the nonlinear dynamics being important in the system, and, thus, using the linear equation for the description of the plasma is not correct. However, if this transition stage is short, and then evolves back into either a linear stage or the third-order term becomes dominant, we can still predict the behavior of the system.

As was mentioned earlier, the fourth-order term can contribute to the saturation of the instability. Since we are not identifying the mechanism of the saturation, it is not necessary to use the fourth-order term in the present analysis. However, to understand the full dynamics of the instability, the fourth-order term becomes very important and needs to be accounted for [19].

5. Explosive instabilities in stretched magnetic-field topology

To demonstrate the investigation of a possible explosive instability, we decided to use a configuration of the magnetic field that would be similar to the Earth's magnetosphere in the slightly excited state on the night side just prior to the explosive release of energy in substorm intensifications (Fig. 1) [14]. Although the plasma in the near-Earth plasmasheet is high β, we used a cold-plasma approximation since an initial equilibrium ($\boldsymbol{j} \times \boldsymbol{B} = 0$) is easier to find and we shall be using this example only for illustrative purposes. The topology of a force-free field is more complicated than the hot plasma case in Earth's magnetosphere. In our configuration, we stretch the field so that the projection of a field

Fig. 1. Comparison between a dipolar topology and the stretched field line topology we used. Field line $L = 10$ is stretched to 12Re at the equator. Numbers denote corresponding field lines. This is a projection of the stretched field lines on the meridional plane. In the force-free configuration these field lines have an azimuthal component.

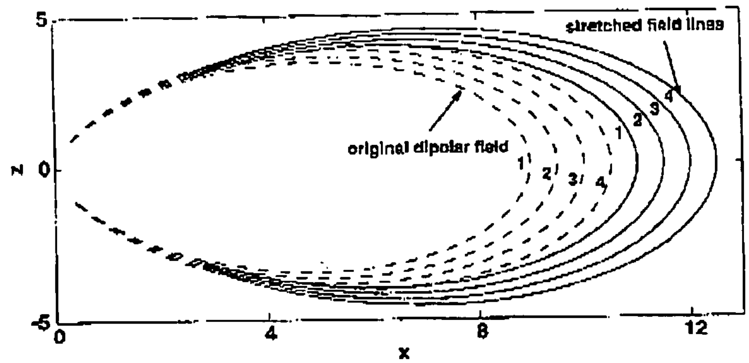
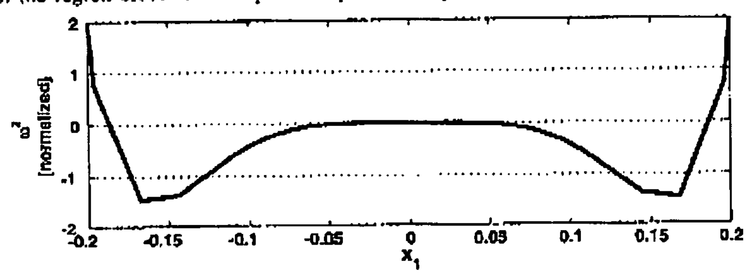


Fig. 2. Dependence of ω^2 on the x_1 coordinate (along field lines). The frequency is normalized to the order of 1. Clearly, in the region close to the equatorial plane the system is unstable.

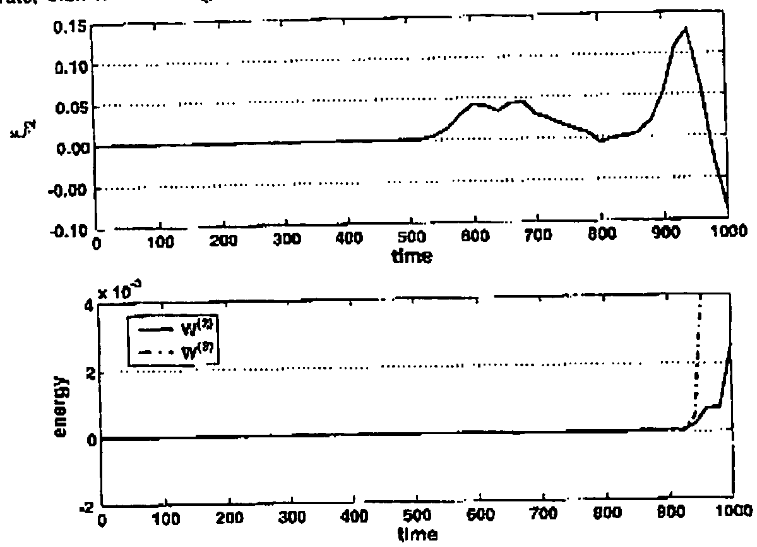


line on the meridional plane has a configuration similar to those shown in ref. 14. Since the condition $j \times B = 0$ requires the current to be parallel to the magnetic field, the magnetic-field line will not be in meridional plane, but must have an azimuthal component. The cold-plasma equilibrium can be found by the use a connection between the magnetic-field geometry and coordinates ($e_1 = B/B$). In other words, the magnetic field can be derived from the coordinates as $B = M/g_{11}$. Therefore, for a cold plasma, it is possible to define the coordinates first, and then simply calculate the magnetic field. From the conservation of magnetic flux ($\nabla \cdot B = 0$) it follows that the relation $g_{11} = g_{22}g_{33}$ must hold for the coordinates used. Then we solve (40)–(42). Solutions of these equations are then substituted into the expressions for the second- and third-order potential energy. The solution of (40) is trivial, thus reducing the problem to two dimensions.

Assuming that the ambient magnetic field changes on a larger spatial scale than the plasma displacement, the dispersion relation for this system is approximated by (44), with the Alfvén velocity defined as $V_A^2 = B^2/\rho$. k_1 and k_2 are the components of the wave vector. Figure 2 shows an example of the dispersion relation (44) in the case of equal wave length in both the radial direction and along field lines. This dispersion relation allows instability in regions close to the equatorial plane. However, due to the scaling of the problem, it is more reasonable to expect the radial wavelength to be shorter than the wavelength along a field line. In such a case the area of the instability can be greatly enhanced.

For the numerical solution of (41) and (42), we used the method outlined in ref. 20. We used normalized units with the unit of length equal to 1 Earth radius. Time was normalized so that Alfvén

Fig. 3. Dynamics of the system in the equatorial plane (top), and the second- and third-order energy densities (bottom). Time is in nondimensional units. The linear instability features two stages, at first it appears to saturate, then it restarts again, and eventually the energy shows onset of explosive behavior.



velocity in the equatorial plane would be 0.01. The magnetic moment was 0.05. The grid has 256 (along field lines) × 128 (in radial direction) grid points. The time step $dt = 0.00001$ and we ran for $0 < t < 1000$ which allowed sufficient time for possible interesting features to develop. The ambient magnetic field and plasma density were defined as $B = M/h_1$, where we put $M = 0.05$ and $\rho = \rho_0/\sin^8\theta$, with $\rho_0 = 10^{-5}$. We had to use such a small value for the magnitude of the ambient magnetic field due to numerical stability problems. The azimuthal wave number was $k_\phi = 2$. The ionospheric boundary was put at 4 (Earth radii) to avoid problems with significant gradients at the boundary. The initial plasma displacement and velocity were

$$\xi_{i0} = 0, \qquad i = \mu, \nu, \phi \tag{49}$$

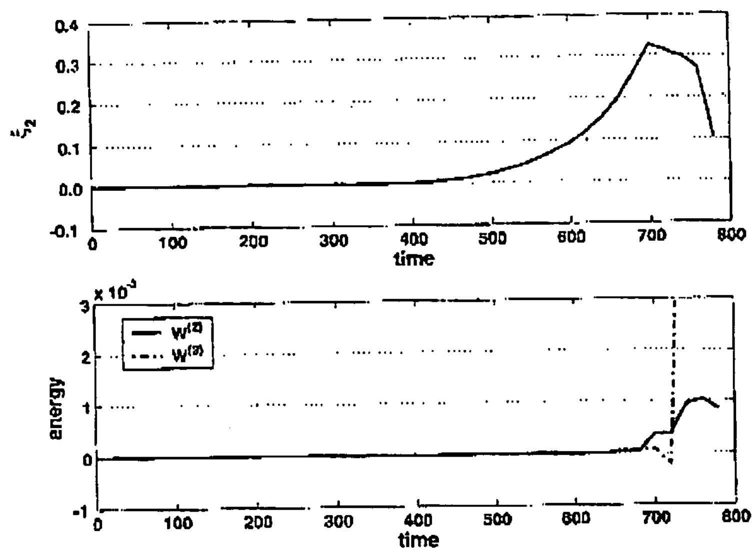
$$u_{i0} = 0, \qquad i = \mu, \phi \tag{50}$$

$$u_{\nu 0} = \Lambda \exp\left(-\frac{(r - r_0)^2}{\delta^2}\right)\exp\left(-\frac{(\theta - \theta_0)^2}{\sigma^2}\right) \tag{51}$$

where we put $\Lambda = 0.0005$, $r_0 = 13$, $\delta = 0.2$, $\theta_0 = \pi/2$, and $\sigma = 0.015$. This models the initial perturbation of the initial equilibrium as a very small impulse in the radial direction. We varied the topology of the field from dipolar to various levels of stretching. As expected, test showed the dipolar configuration to be stable.

Two examples of the instability are shown in Figs. 3 and 4. The top of each figure shows the dynamics of the system in the equatorial plane. In both cases linear instabilities are observed. The bottom of the figures shows the behavior of the energy densities, using the expansion of the Lagrangian (15) calculated using the results of the linear equations (40)–(42). We can observe that at the time corresponding to the saturation of the linear instability the third-order term becomes dominant. This indicates a break down of the linear approximation and the onset of an explosive phase.

Fig. 4. Dynamics of the system in the equatorial plane (top), and the second- and third-order energy densities (bottom). Time is in nondimensional units. After short linear stage the system becomes explosively unstable.



6. Discussion

6.1. Instabilities in cold plasmas

A question can arise concerning the nature of the instabilities in cold plasmas like the model considered here. At first we need to look at the source of possible free energy that can drive the instability. Such a source of energy is the parallel currents $J_\parallel$ that produce the stretching (and twisting to give an azimuthal component) of the magnetic-field lines in this model of cold plasmas. In fact, it is this energy connected to the stretching that is the primary source for the instabilities even in case of high-beta plasmas as was demonstrated in ref. 21.

In the case of cold plasmas, the equilibrium condition is simply $\boldsymbol{J} \times \boldsymbol{B} = 0$. This expression can be alternatively written for ideal MHD as $\nabla \boldsymbol{B}^2/2 = \boldsymbol{B} \cdot \nabla \boldsymbol{B}$ by using the fact that $\boldsymbol{J} = \nabla \times \boldsymbol{B}$. This condition can be compared with the equilibrium condition for a warm plasma, $\nabla(P + \boldsymbol{B}^2/2) = \boldsymbol{B} \cdot \nabla \boldsymbol{B}$. In this case, perturbations of the equilibrium between the total pressure (combined plasma and magnetic pressure) and the curvature can generate a ballooning instability in the plasma. An excellent discussion of Rayley–Taylor instabilities in space plasmas is given in ref. 22. It is reasonable to assume from the similarity of the equilibrium conditions between cold and hot plasmas, where in the former case the magnetic pressure takes the role of the total pressure, that the nature of the instability in a cold plasma will be effectively the same as in case of a hot plasma with the magnetic pressure gradient acting against the field line curvature to generate ballooning. That way it is possible to have an instability even in the absence of a thermal pressure gradient in a plasma.

6.2. Topology of the instability

In a force-free magnetic field, the equilibrium condition $\boldsymbol{j} \times \boldsymbol{B} = 0$ requires the current to be parallel to the magnetic field. This leads to the existence of the azimuthal component of the magnetic field to allow for stretching of the field lines in the meridional plane. Thus, the magnetic-field line is not entirely in one plane, but is twisted across the meridional planes.

If we allow for a small displacement of the field line δB Earth-ward, a current is generated in the azimuthal direction. This current combined with the ambient $j_\parallel$ generates a magnetic-field component eliminating the azimuthal component of the ambient field. Reduction of the azimuthal component of the field causes in turn the generation of current feeding the Earth-ward displacement of the field lines. This way the instability evolves, and eventually brings the field back to a dipolar state. The energy that was stored in enhanced currents is transformed into the kinetic energy of the enhanced flows in a plasma.

7. Conclusion

This paper has addressed several objectives. First, we have shown that the use of geometrical methods such as component formalism, transformation relations, and the formulas for determinants and inverse matrices can simplify the description of magnetized plasmas. Compared with a traditional vector approach, the derivations are more transparent, fully self-consistent, and the results are in a compact form due to the easy reduction of redundant terms. Also, in systems with special symmetries, these symmetries are relatively easy to incorporate into the equations.

We have applied this method to calculate the expansion of the Lagrangian within ideal MHD, but the same method can be applied to nonideal MHD models by adding additional terms to the unperturbed Lagrangian, or alternatively, the method can also be used for other approaches to plasmas with a known Lagrangian. We also present suggestions for the use of this formalism in plasma problems that are complicated by nontrivial topologies of the magnetic field, such as calculation of eigenmodes in asymmetric topologies, or investigations of nonlinear instabilities in curved magnetic-field topologies.

Treating plasma perturbations as a coordinate change, a method common for elasticity theory [23], we expanded the Lagrangian of ideal MHD up to the third order in plasma displacement. The first two orders of the expanded Lagrangian yield equilibrium conditions and the general linear wave equation for that model. We have shown that for ideal MHD this equation is consistent with the results obtained by the direct linearization of the MHD equations using a traditional vector approach. We have also derived dispersion relations and a linear instability criterion for a cold plasma in a general curvilinear field. The third-order term provides information about nonlinear plasma stability. In systems close to equilibrium, we can use the results of the linear model to calculate the second- and third-order energy densities. Their relative size provides insight into the initial stages of nonlinear dynamics. Specifically, when the third-order energy prevails, the system has evolved into an explosively unstable stage.

Last, we demonstrated this method in the investigation of a cold plasma in a curved, stretched magnetic-field topology where the results reveal the possibility of explosive behavior that follows an initial linear instability. In our study with a cold plasma model, the energy source for the instability is contained in the stretching of the field lines beyond the dipolar configuration and the current enhancement connected with this stretching. This two-stage instability is consistent with the results of nonlinear MHD modeling [24]. This qualitative agreement suggests that this method is indeed an effective tool for the identification of the onset of explosive instability.

Acknowledgments

This research has been funded by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

References

1. S. Ortolani and D.D. Schnack. Magnetohydrodynamics of plasma relaxation. 1st ed. World Scientific, Singapore. 1993.
2. S.I. Itoh, K. Itoh, H. Zushi, and A. Fukujama. Plasma Phys. Control. Fusion, 40, 879 (1998).
3. S.C. Cowley and M. Artun. Phys. Rep. 283, 185 (1997).
4. E. Friedrich, J.C. Samson, and I. Voronkov. Phys. Plasmas, 8, 1104 (2001).

5. O.A. Hurricane, B.H. Fong, and S.C. Cowley. Phys. Plasmas, 4, 3565 (1997).
6. I.B. Bernstein, E.A. Frieman, M.D. Kruskal, and R.M. Kulsrud. Proc. R. Soc. London A, 17, 244 (1958).
7. V.I. Ilginois and V.P. Pastukhov. JETP Lett. 72, 530 (2000).
8. D. Pfirsch and R.N. Sudan. Phys. Fluids, 7, 2052 (1993).
9. H. Flanders. Differential forms with application to the physical sciences. Academic Press, New York – London. 1963.
10. C.W. Misner, K.S. Thorne, and J.A. Wheeler. Gravitation. W.H. Freeman & Co., New York. 1973.
11. B.F. Schutz. Geometrical methods of mathematical physics. Cambridge University Press, Cambridge, New York. 1980.
12. B.G. Harrold and J.C. Samson. Geophys. Res. Lett. 19, 1811 (1992).
13. W.W. Liu. J. Geophys. Res. 102, 4927 (1997).
14. J.A. Wanliss, J.C. Samson, and E. Friedrich. J. Geophys. Res. 105, 27673 (2000).
15. J.E. Marsden and T.S. Ratiu. Introduction to mechanics and symmetry. Springer-Verlag, New York, Berlin. 1999.
16. P.J. Morrison and J.M. Greene. Phys. Rev. Lett. 45, 790 (1980).
17. H. Cendra, D.D. Holm, M.J.W. Hoyle, and J.E. Marsden. J. Math. Phys. 39, 3138 (1998).
18. S. Lundquist. Phys. Rev. 83, 307 (1951).
19. B.H. Fong, S.C. Cowley, and O.A. Hurricane. Phys. Rev. Lett. 82, 4651 (1999).
20. G.J. Rickard and A.N. Wright. J. Geophys. Res. 99, 13455 (1994).
21. I.O. Voronkov. Ph.D. thesis, University of Alberta, 1998.
22. S.I. Ohtani and T. Tamao. J. Geophys. Res. 98, 19369 (1993.)
23. L.D. Landau and E.M Lifshitz. Theory of elasticity. Pergamon Press, New York. 1986.
24. I.O. Voronkov, J.C. Samson, E. Friedrich, R. Rankin, V.T. Tikhonchuk, and E.F. Donovan. *In* Proceedings of the 5th International Conference on Substorms, 16–20 May 2000, St. Petersburg, Russia. European Space Agency, SP series, (SP-443). 2000. p. 249.

Notation

B	ambient magnetic field
p	ambient pressure
v	velocity
ρ	plasma density
γ	isotropic coefficient
g_{ij}	metric tensor
g	determinant of the metric tensor g_{ij}
ϵ_{ijk}	fully anti-symmetric unit tensor
δ^{ijk}_{lmn}	generalized Kronecker delta (product of δs antisymmetrized in all indices)
$a_{[ij]}$	antisymmetrization in i, j indices
$a_{,i}$	$= \partial_i a$
$a_i b^i$	$\equiv \sum_i a_i b^i$